

NONNEGATIVE TRIGONOMETRIC POLYNOMIALS AND A ZERO-FREE REGION FOR THE RIEMANN ZETA-FUNCTION

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ABSTRACT. We prove that the Riemann zeta-function $\zeta(\sigma + it)$ has no zeros in the region $\sigma \geq 1 - 1/(5.573412 \log|t|)$ for $|t| \geq 2$. This represents the largest known zero-free region within the critical strip for $3.06 \cdot 10^{10} < |t| < \exp(10151.5)$. Our improvements result from determining some favorable trigonometric polynomials having particular properties, and from analyzing the error term in the method of Kadiri. We also improve an upper bound in a question of Landau regarding nonnegative trigonometric polynomials.

1. INTRODUCTION

Let $\zeta(s)$ denote the Riemann zeta-function, where throughout this article we write $s = \sigma + it$, with σ and t real numbers. It is well known that $\zeta(s)$ is zero at each negative even integer—these are the trivial zeros of the zeta-function—and that all nontrivial zeros of this function occur in the critical strip in the complex plane, where $0 < \sigma < 1$. Further, the nontrivial zeros are symmetric about the line $\sigma = 1/2$. The Riemann hypothesis states that all nontrivial zeros are on this line.

Determining zero-free regions within the critical strip has long been of great interest in number theory, since such results bear directly on questions regarding the distribution of prime numbers. For example, the prime number theorem was established in 1896 by Hadamard and de la Vallée Poussin by proving that $\zeta(s)$ has no zeros on the line $\sigma = 1$. Moreover, de la Vallée Poussin established in 1899 [2] that $\zeta(s) \neq 0$ in a region of the form

$$\sigma > 1 - \frac{1}{R_0 \log|t|},$$

where R_0 is a particular constant and $|t|$ is sufficiently large. We refer to a region of this form as a classical zero-free region of the Riemann zeta-function.

An improved zero-free region for $\zeta(s)$ was established by Korobov [7] and Vinogradov [20] in 1958, who showed independently that $\zeta(s) \neq 0$ for

$$\sigma > 1 - \frac{1}{R_1(\log|t|)^{2/3}(\log \log|t|)^{1/3}}$$

for some constant R_1 . Ford [4] established an explicit region of this type, showing that one can take $R_1 = 57.54$ for $|t| \geq 3$. Clearly, Ford's region is asymptotically

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TABLE 1. Improvements in the constant R_0 in the classical region.

		R_0	b_1	b_2
1899	de la Vallée Poussin [2]	30.4679	1	—
1938	Westphal [22]	17.537	1	0.25
1962	Rosser & Schoenfeld [14]	17.51631	1	0.3
1970	Stechkin [18]	9.65	0.91	0.28
1975	Rosser & Schoenfeld [15]	9.64591	1	0.3
1977	Kondrat'ev [6]	9.547897	(see Table 2)	
2002	Ford [4]	8.463	0.9	0.225
2005	Kadiri [5]	5.69693	0.91	0.265

superior to the classical one, but for values of $|t|$ that are not too large, explicit forms of the classical region provide better bounds on σ than Ford's result. Because of this, explicit forms of the classical zero-free region remain of interest, and find application in number theory: see for instance [3, 12].

Table 1 exhibits the history of improvements in the value of R_0 in the classical region, beginning with de la Vallée Poussin's value, $R_0 = 30.4679$, from 1899, through to the 2005 result of Kadiri [5], who established that $R_0 = 5.69693$ is permissible. Each of these works employed an even trigonometric polynomial $f(\varphi) = \sum_{k=0}^n a_k \cos(k\varphi)$ with real coefficients and possessing certain properties: $f(\varphi) \geq 0$ for all real φ , each coefficient a_k is nonnegative, and $a_1 > a_0$. For example, de la Vallée Poussin used the polynomial $(1 + \cos \varphi)^2 = \frac{3}{2} + 2 \cos \varphi + \cos 2\varphi$. After this result, each of the other efforts summarized in this table employed a polynomial of degree $n = 4$, except for Kondrat'ev, who employed one of degree 8, described below. The trigonometric polynomial used in each of the other results summarized in Table 1 is indicated by the values of b_1 and b_2 in the last two columns: these designate the nonnegative even trigonometric polynomial $(b_1 + \cos \varphi)^2 (b_2 + \cos \varphi)^2$.

Kondrat'ev's work was motivated by a result of Landau from 1908 [8]. Let P_n denote the set of even trigonometric polynomials $f_n(\varphi) = \sum_{k=0}^n a_k \cos(k\varphi)$ having nonnegative real coefficients with $a_1 > a_0$ and satisfying $f_n(\varphi) \geq 0$ for all real φ . Define the real number V_n by

$$V_n = \inf_{f_n \in P_n} \frac{f_n(0) - a_0}{(\sqrt{a_1} - \sqrt{a_0})^2}. \quad (1.1)$$

For convenience, we let $A = A(f_n) := f_n(0) - a_0$. Certainly V_n is decreasing in n , so we define

$$V = \lim_{n \rightarrow \infty} V_n. \quad (1.2)$$

Landau proved that for $\epsilon > 0$ one may take

$$R_0 = \frac{V}{2} + \epsilon$$

for $|t| > T(\epsilon)$ in the classical zero-free region. Stechkin [18] improved this in 1970, showing that one may select

$$R_0 = \frac{V}{2} \left(1 - \frac{1}{\sqrt{5}} \right) + \epsilon$$

for $|t| > T(\epsilon)$, and by using the degree 4 trigonometric polynomial shown in this entry in Table 1, Stechkin established his value of $R_0 = 9.65$ for $|t| > 12$.

TABLE 2. Coefficients of the polynomial of Kondrat'ev, $k_8(\varphi)$.

a_0	1
a_1	1.733792817542616
a_2	1.110484293773627
a_3	0.4895739485699287
a_4	0.1180328991868943
a_5	$7.549474144412732 \cdot 10^{-9}$
a_6	$7.994175811779779 \cdot 10^{-10}$
a_7	0.009253861629263798
a_8	0.004429241403972788
A	3.465567070455195

Kondrat'ev [6] employed a computational strategy to search for trigonometric polynomials $f_n \in P_n$ that would produce improved bounds on V_n , and thus on R_0 . While the details of his method of optimization are not recorded in his 1977 article, his search at $n = 8$ produced the polynomial $k_8(\varphi)$, whose normalized coefficients are listed in Table 2.

Kadiri [5] employed a more complicated analysis to obtain the value $R_0 = 5.69693$. We review her method in Section 2. In this article, we obtain an improved value for R_0 by using Kadiri's method, with two main improvements. First, we construct an improved trigonometric polynomial of degree $n = 16$, using the randomized optimization method of simulated annealing to search for favorable polynomials. Second, we analyze the error term of this method and obtain some savings by employing additional computations. We prove the following theorem.

Theorem 1. *There are no zeros of $\zeta(\sigma + it)$ for $|t| \geq 2$ and*

$$\sigma > 1 - \frac{1}{5.573412 \log |t|}.$$

This represents the largest known zero-free region for the zeta-function within the critical strip for $3.06 \cdot 10^{10} < |t| < \exp(10151.5)$.

Our work also allows us to improve a bound on Landau's quantity V from (1.2). Exact values for V_n are known only for $n \leq 6$; for the general case, Arestov and Kondrat'ev proved in 1990 [1] that

$$34.468305 < V < 34.5035864. \quad (1.3)$$

More information on what is known regarding values of V_n and similar quantities may be found in the survey of Révész [13]. In this article, we determine improved values for V_n for a number of values of n , and reduce the gap in (1.3) by approximately 40%.

Theorem 2. *Let P_n denote the set of even trigonometric polynomials $f_n(\varphi) = \sum_{k=0}^n a_k \cos(k\varphi)$ of degree n having each $a_k \geq 0$ and $a_1 > a_0$, and let*

$$V = \lim_{n \rightarrow \infty} \inf_{f_n \in P_n} \frac{f_n(0) - a_0}{(\sqrt{a_1} - \sqrt{a_0})^2}.$$

Then $34.468305 < V < 34.4889920009$.

In Section 2, we summarize Kadiri's method, and we describe some improvements we make to this method in Section 3. Section 4 provides details on our

bounding of the error term in Kadiri's method. Section 5 describes our methods for searching for improved trigonometric polynomials by using simulated annealing. Section 6 reports on the results of our searches, describes the reduction in the gap of permissible values for V_n for a number of integers n , and establishes Theorems 1 and 2. It also investigates one possible avenue for further improvements. Finally, Section 7 briefly discusses some applications of these results.

2. KADIRI'S METHOD

We summarize the method of Kadiri [5] for reducing the constant R_0 in the classical zero-free region of the Riemann zeta-function. We use the same notation. This strategy relies on a number of parameters. First, one requires a trigonometric polynomial $f_n \in P_n$. Next, we let T_0 be a height to which the Riemann hypothesis has been verified, and we let R be a positive constant for which the classical zero-free region of the Riemann zeta-function has already been established. Also, we let $r < R$ be a positive constant, and we let t_0 denote a constant in $[1, T_0]$ that may be selected later.

We suppose that $\zeta(s)$ has a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\gamma_0 > 0$ which lies just outside an established zero-free region of the zeta-function, so that

$$1 - \frac{1}{r \log \gamma_0} \leq \beta_0 \leq 1 - \frac{1}{R \log \gamma_0}.$$

We aim to show that β_0 in fact satisfies a stronger inequality of the form

$$\beta_0 \leq 1 - \frac{1}{R_0 \log \gamma_0}$$

for some $R_0 \in (r, R)$. The parameter r thus represents a lower bound on what we might achieve.

We now define

$$\begin{aligned} \eta &= \frac{1}{r \log \gamma_0}, & \sigma &= 1 - \frac{1}{R \log(n\gamma_0 + t_0)}, & w &= \frac{1 - \sigma}{\eta}, \\ \eta_0 &= \frac{1}{r \log T_0}, & \sigma_0 &= 1 - \frac{1}{R \log(nT_0 + t_0)}, & w_0 &= \frac{1 - \sigma_0}{\eta_0}, \end{aligned} \tag{2.1}$$

and, for any fixed $\theta \in (\pi/2, \pi)$, we define the function $h_\theta(u)$ by

$$\begin{aligned} h_\theta(u) = \sec^2 \theta \left\{ \sec^2 \theta \left(\frac{-\theta}{\tan \theta} - \frac{u}{2} \right) \cos(u \tan \theta) - \frac{2\theta}{\tan \theta} - u \right. \\ \left. - \frac{\sin(2\theta + u \tan \theta)}{\sin 2\theta} + 2 \left(1 + \frac{\sin(\theta + u \tan \theta)}{\sin \theta} \right) \right\}. \end{aligned}$$

We then set $g_1(\theta) = h_\theta(0)$, so that

$$g_1(\theta) = \sec^2(\theta) (3 - \theta \tan \theta - 3\theta \cot \theta),$$

and we let

$$d_1(\theta) = -\frac{2\theta}{\tan \theta}$$

and

$$m(\theta) = \max \{ |h''_\theta(u)| : 0 \leq u \leq d_1(\theta) \}.$$

Next, we set δ to be the solution in $[0, 1]$ to the equation $\kappa_2(\delta) = \kappa_3(\delta)$, where

$$\kappa_2(\delta) = \kappa_2(\delta, \theta) = \frac{g_1(\theta)(2\sigma_0 - 1) - \frac{m(\theta)\eta_0^2}{2\sigma_0 - 1}}{(1 + 2\delta)g_1(\theta) + \left(\frac{1}{\delta} + \frac{1}{\delta + 2\sigma_0 - 1}\right)m(\theta)\eta_0^2}$$

and

$$\kappa_3(\delta) = \kappa_3(\delta, \theta) = \frac{g_1(\theta)(2\sigma_0 - 1) - \frac{m(\theta)\eta_0^2}{2\sigma_0 - 1}}{\left(\frac{1}{\delta} + \frac{1+\delta}{(\delta+2\sigma_0-1)^2}\right)g_1(\theta) + \left(\frac{1}{\delta^3} + \frac{1}{(\delta+2\sigma_0-1)^3}\right)m(\theta)\eta_0^2},$$

noting the additional requirements that

$$\frac{\sqrt{5}-1}{2} \leq \delta \leq 0.866$$

and

$$\frac{1}{\delta} + \frac{1}{0.99 + \delta} \leq \frac{1}{\kappa} \leq \frac{1}{\delta^3} + \frac{1}{(1 + \delta)^3}.$$

In fact, one may select $0 \leq \kappa \leq \min\{\kappa_2(\delta), \kappa_3(\delta)\}$, and $\delta \geq \delta'$ if δ' is the solution in $[0, 1]$ of $\kappa_2(\delta) = \kappa_3(\delta)$, but we find that the described selections for δ and κ satisfy the required constraints in our applications.

Finally, let

$$K(w, \theta) = \int_0^{d_1(\theta)} (a_1 e^{-u} - a_0) h_\theta(u) e^{wu} du,$$

where a_0 and a_1 are the coefficients of the first two terms of the selected trigonometric polynomial $f_n \in P_n$. It then follows from the arguments given in [5] that

$$R_0 \leq \frac{Ag_1(\theta)(1 - \kappa)}{2(K(w, \theta) - C(\eta))}, \quad (2.2)$$

where $C(\eta)$ is a small error term, and we recall that $A = f_n(0) - a_0$. Kadiri proved that $K(w, \theta)$ is increasing for $w \geq w_0$ for each fixed $\theta \in (\pi/2, \pi)$, and that $C(\eta)$ is decreasing and nonpositive for $0 \leq \eta \leq \eta_0$. It follows that

$$R_0 \leq \frac{Ag_1(\theta)(1 - \kappa)}{2K(w_0, \theta)} \quad (2.3)$$

for any fixed $\theta \in (\pi/2, \pi)$. Kadiri selected Rosser and Schoenfeld's value [15] for R , so $R = 9.645908801$, and chose $r = 5$, $T_0 = 3\,330\,657\,430.697$ (as established in [21]), $t_0 = 10$, $\theta = 1.848$, and the trigonometric polynomial of degree $n = 4$ shown in Table 1. Kadiri then obtained the value 5.69693 by using an iterative procedure.

First, one computes an initial improvement in R_0 from (2.3), and then one replaces R by this R_0 , and increases the value of r as well, and performs the computation again. Note that changing R and r in the method alters the values of η_0 , σ_0 , and w_0 , and hence the values of κ and $K(w_0, \theta)$, and so a new value for R_0 is obtained from (2.3). The value chosen for r must remain smaller than any improved value for R_0 . This procedure is repeated until the r and R_0 values lie within a tolerance of $\Delta = 10^{-5}$, and then this iteration is repeated, with r reset to 5 and R set to the latest value of R_0 . This outer iteration is then continued until R_0 no longer improves by more than $v = 5 \cdot 10^{-6}$. Using Kadiri's trigonometric polynomial and the selected values for T_0 , t_0 and θ , after six iterations this method produces the value $R_0 = 5.696924085 \dots$.

3. IMPROVEMENTS

Our improvement in the value of R_0 over Kadiri's result has three main sources. First, we employ a new trigonometric polynomial. Second, we save some information in the error term $C(\eta)$ to use in the iterative calculation. Third, we adjust some of the other parameters in kind, and exploit some more recent knowledge about the zeros of the zeta-function.

We employ the trigonometric polynomial $F_{16}(\varphi) = \sum_{k=0}^{16} a_k \cos(k\varphi)$, whose coefficients are listed in Table 5 in Section 6. A graph of this polynomial over $(\pi/2, \pi)$ is shown in Figure 4. The method used to construct this polynomial is described in Section 5, and this method guarantees that $F_{16}(\varphi) \geq 0$ for all φ . It is evident from Table 5 that each $a_k \geq 0$ and $a_1 > a_0$, so $F_{16} \in V_{16}$.

To obtain some savings from the error term $C(\eta)$, we consider a new parameter η_1 with $0 < \eta_1 \leq \eta_0$. When $\eta \leq \eta_0$, we have $\gamma_0 \geq \exp(1/r\eta_1)$ by (2.1), and since $\log x / \log(nx + t_0)$ is increasing in x for fixed n and t_0 , it follows that

$$w = \frac{r \log \gamma_0}{R \log(n\gamma_0 + t_0)} \geq \frac{1/R\eta_1}{\log(n \exp(1/r\eta_1) + t_0)} =: w_1.$$

Thus, for $\eta \leq \eta_1$ we may bound the denominator in (2.2) below by $K(w_1, \theta)$.

Since $C(\eta)$ is decreasing on $[0, \eta_0)$, we have $C(\eta) \leq C(\eta_1)$ for $\eta_1 \leq \eta \leq \eta_0$, so in this range the denominator in (2.2) is bounded below by $K(w_0, \theta) - C(\eta_1)$. Therefore, we can replace the bound (2.3) by

$$R_0 \leq \frac{A}{2} g_1(\theta)(1 - \kappa) \left(\min_{0 < \eta_1 \leq \eta_0} \max \left\{ \frac{1}{K(w_1, \theta)}, \frac{1}{K(w_0, \theta) - C(\eta_1)} \right\} \right). \quad (3.1)$$

The function $C(\eta)$ is described in detail in Section 4.

We select $T_0 = 3.06 \cdot 10^{10}$ as established in [10], $t_0 = 10^5$, $\theta = 1.85573$, $r = 5$, and $R = 5.7$. After each iteration, we reset r to the average of its current value and the value of R_0 just obtained, and continue this inner process until the distance between these two values is less than $\Delta = 10^{-6}$. The outer iteration is continued until R_0 no longer improves by more than $v = 5 \cdot 10^{-7}$. In addition, in each computation of R_0 a binary search is employed to determine a value for $\eta_1 \in (0, \eta_0]$: we search for a value of η_1 in this range where $K(w_1, \theta)$ and $K(w_0, \theta) - C(\eta_1)$ differ by no more than $\epsilon = 10^{-3}$. Table 3 lists the values of several parameters in the computation at the end of each inner iteration, so just before r is reset to 5 and R_0 is assigned to R . This table exhibits the values for η_0 , η_1 , κ , and δ in each iteration as well. We obtain the value $R_0 = 5.5734118005\dots$ after seven rounds; an eighth round produces $R_0 = 5.57341178\dots$

We remark that using the new trigonometric polynomial, combined with the strategy of saving some information in the error term, accounts for approximately 93.6% of the improvement in the value of R_0 obtained here. The remaining 6.4% is due to the larger value of T_0 that we employ.

4. ANALYZING THE ERROR TERM

We describe the error term $C(\eta)$ from (2.2) in some detail, highlighting aspects where our estimates differ from those used in [5]. Following Kadiri, we write $C(\eta) = C_1(\eta) + C_2(\eta) + C_3(\eta) + C_4(\eta)$, and describe each of these summands in turn. First,

TABLE 3. Values of parameters in successive runs of the outer iteration. All values are rounded.

R	r	$\eta_0 \cdot 10^3$	$\eta_1 \cdot 10^3$	κ	δ	R_0
5.7000000	5.58682	7.41347	0.861315	0.440100	0.620251	5.5868212
5.5868212	5.57486	7.42938	0.876546	0.439964	0.620293	5.5748558
5.5748558	5.57357	7.43109	0.878187	0.439949	0.620298	5.5735676
5.5735676	5.57343	7.43128	0.878364	0.439948	0.620298	5.5734286
5.5734286	5.57341	7.43130	0.878383	0.439948	0.620298	5.5734136
5.5734136	5.57341	7.43130	0.878385	0.439948	0.620298	5.5734120
5.5734120	5.57341	7.43130	0.878386	0.439948	0.620298	5.5734118

equations (55) and (56) in [5] produce $C_1(\eta) = \eta g_1(\theta) \sum_{k=0}^n a_k c_1(k)$, where

$$c_1(0) = \frac{\kappa - 1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{3}{2} \right) - \frac{\kappa}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma_0 + \delta}{2} + 1 \right)$$

and

$$c_1(k) = \frac{\kappa - 1}{2} \log \frac{2\pi}{k} + \frac{1}{2} \min\{r_2(\sigma_0 + 2, 3, kT_0), r_3(\sigma_0 + 2, 3, kT_0)\}$$

for $1 \leq k \leq n$, where

$$r_2(x_0, x_1, y_0) = \frac{1 - \kappa}{2} \log \left\{ 1 + \frac{(x_1 + \delta)^2}{y_0^2} \right\} + \frac{1}{y_0} \left(\tan^{-1} \frac{y_0}{x_1} + \kappa \tan^{-1} \frac{y_0}{x_1 + \delta} \right),$$

$$r_3(x_0, x_1, y_0) = \frac{1}{3y_0} \left(\frac{1}{x_0} + \frac{\kappa}{x_0 + \delta} \right) + \frac{x_1^2 + \kappa(x_1 + \delta)^2}{2y_0^2}.$$

Second, from equation (60) in [5] we have $C_2(\eta) = q_1\eta + q_2\eta^2 + q_3\eta^3$, where

$$q_1 = -\kappa g_1(\theta) \left\{ \frac{a_0}{\delta} + \frac{\sigma_0 - 1 + \delta}{2} \sum_{k=1}^n \frac{a_k}{(kT_0)^2} \right\},$$

$$q_2 = M^*(-r/R, \theta) \sum_{k=1}^n \frac{a_k}{(kT_0)^2},$$

$$q_3 = \frac{a_0 m(\theta) \kappa}{(\sigma_0 - 1 + \delta)^3} + \frac{m(\theta) \kappa}{\sigma_0 - 1 + \delta} \sum_{k=1}^n \frac{a_k}{(kT_0)^2}.$$

Here, $M^*(z, \theta)$ is an upper bound on

$$M(z, \theta) = \int_0^{d_1(\theta)} |h''_\theta(u)| e^{-zu} du.$$

To compute a value for $M^*(z, \theta)$ in an efficient way when z varies and θ is fixed, we compute each of the integrals

$$M_k(\theta) = \int_0^{d_1(\theta)} |h''_\theta(u)| u^k du$$

once at the start of the computation, and then the fact that $e^y \leq 1 + y + y^2/2 + y^3/3.45$ for $0 \leq y \leq 1.91094 \dots$ implies that we may take

$$M^*(z, \theta) = M_0(\theta) - M_1(\theta)z + \frac{M_2(\theta)z^2}{2} - \frac{M_3(\theta)z^3}{3.45}$$

with $z = -r/R$ since $0 < r < R$, provided that $d_1(\theta) \leq y_0$. This last condition is satisfied for $1.8469 \dots < \theta < \pi$, and the values of θ that we employ lie within this range. Naturally, we must take care to round $M_0(\theta)$ and $M_2(\theta)$ up at the last decimal place of the calculated precision, and $M_1(\theta)$ and $M_3(\theta)$ down. (This is similar to the analysis in [5], but here we add the z^3 term for a sharper estimate.)

Third, equations (52), (53), and (54) in [5] produce $C_3(\eta) = p_1\eta + p_2\eta^2 + p_3\eta^3$, where

$$\begin{aligned} p_1 &= a_1 g_1(\theta) \left\{ \left(\frac{1}{\delta} + \frac{1}{\sigma_0 - \eta_0 + \delta} \right) \kappa - 1 \right\} + M^*(-r/R, \theta) \eta_0 \sum_{k=0}^n a_k c_{30}(kT_0, t_0), \\ p_2 &= \frac{(1 + 2\kappa)m(\theta)\eta_0}{\sigma_0 - \frac{1}{2}} \sum_{k=0}^n a_k c_{30}(kT_0, t_0), \\ p_3 &= a_1 m(\theta) \left\{ \left(\frac{1}{\delta^3} + \frac{1}{(\sigma_0 - \eta_0 + \delta)^3} \right) \kappa + 1 \right\}, \end{aligned}$$

and where $c_{30}(t, t_0)$ is required to be an upper bound on the sum

$$\Sigma(t, t_0) = \sum_{\substack{\zeta(\beta+i\gamma)=0 \\ |\gamma| \geq t+t_0}} \frac{1}{(\gamma - t)^2}. \quad (4.1)$$

We depart from [5] here in obtaining an expression for $c_{30}(t, t_0)$: Kadiri employs an explicit bound on the error term for the function $N(T)$, the number of nontrivial zeros of the Riemann zeta-function with imaginary part in $[0, T]$. We pursue another approach. For $t = 0$, we have

$$\Sigma(0, t_0) = 2 \left(\sum_{\gamma > 0} \frac{1}{\gamma^2} - \sum_{0 < \gamma \leq t_0} \frac{1}{\gamma^2} \right),$$

where all the sums are over the pertinent zeros of the zeta-function in the critical strip. From [16, Lemma 2.9] we have that the first sum in this last expression is at most 0.023105, and we may compute the latter sum explicitly for a reasonable value of t_0 . Using this strategy, we compute that we may take $c_{30}(0, 10^5) = 0.00027$. For $t > 0$, we note that

$$\Sigma(t) = \sum_{\gamma \geq t+t_0} \frac{1}{(\gamma - t)^2} + \sum_{\gamma \geq t+t_0} \frac{1}{(\gamma + t)^2},$$

and we use a result of Lehman [9], which allows us to write

$$\sum_{\gamma > T_1} \phi(\gamma) = \frac{1}{2\pi} \int_{T_1}^{\infty} \phi(x) \log(x/2\pi) dx + \xi \left\{ 4\phi(T_1) \log T_1 + 2 \int_{T_1}^{\infty} \frac{\phi(x)}{x} dx \right\},$$

where ξ denotes a constant of absolute value at most 1, when $\phi(x)$ is continuous, positive, and monotone decreasing on $x > T_1 > 2\pi e$. Using $\phi(x) = 1/(x - t)^2 + 1/(x + t)^2$, and ignoring negligible savings, we find that we may take

$$\begin{aligned} c_{30}(kT_0) &= \frac{1}{2\pi} \int_{t_0}^{\infty} \log \left(\frac{x + kT_0}{2\pi} \right) \left(\frac{1}{x^2} + \frac{1}{(x + 2kT_0)^2} \right) dx \\ &\quad + 4 \log(kT_0 + t_0) \left(\frac{1}{t_0^2} + \frac{1}{(t_0 + 2kT_0)^2} \right) + \frac{4}{kT_0 t_0} \end{aligned}$$

for $1 \leq k \leq n$.

Last, equations (61) and (64) in [5] yield

$$C_4(\eta) = \eta^3 m(\theta) \sum_{k=0}^n a_k \{C_{41}(k) + C_{42}(k)\},$$

where

$$\begin{aligned} C_{41}(k) &= \frac{1}{2\pi(\sigma_0 - \frac{1}{2})} \int_{-\infty}^{\infty} \frac{U_0(t)}{(\sigma_0 - \frac{1}{2})^2 + (kT_0 - t)^2} dt \\ &\quad + \frac{\kappa}{2\pi(\sigma_0 - \frac{1}{2} + \delta)} \int_{-\infty}^{\infty} \frac{U_0(t)}{(\sigma_0 - \frac{1}{2} + \delta)^2 + (kT_0 - t)^2} dt, \\ C_{42}(0) &= \frac{1}{\sigma_0^3} + \frac{\kappa}{(\sigma_0 + \delta)^3}, \quad C_{42}(k) = \left(\frac{1}{\sigma_0} + \frac{\kappa}{\sigma_0 + \delta} \right) \frac{1}{(kT_0)^2}, \quad \text{if } k \geq 1, \end{aligned}$$

and, by Lemma 3.6 of [5],

$$U_0(t) \leq \begin{cases} \frac{1}{2} \log \left(\frac{16}{1+4t^2} \right) + \frac{2}{1+4t^2} + 2, & \text{if } |t| < \frac{1}{2}, \\ \left| \log \frac{|t|}{2} - \frac{2}{1+4t^2} \right| + \frac{2}{3|t|} + \frac{1}{8t^2}, & \text{if } |t| \geq \frac{1}{2}. \end{cases} \quad (4.2)$$

We estimate the first integral in the definition of $C_{41}(k)$ as follows; the second integral is estimated similarly. We write this integral as

$$I = \int_{-\infty}^{\infty} \frac{U_0(t)}{a^2 + (b-t)^2} dt,$$

where $a = \sigma_0 - \frac{1}{2}$ and $b = kT_0$. Note that $\log \frac{|t|}{2} - \frac{2}{1+4t^2} \geq 0$ for $|t| \geq t^* = 2.205\dots$. When $k = 0$ we make use of the symmetry inherent in the integrand to write

$$\int_{-\infty}^{\infty} \frac{U_0(t)}{a^2 + t^2} dt = 2 \left(\int_0^{1/2} \frac{U_0(t)}{a^2 + t^2} dt + \int_{1/2}^{t^*} \frac{U_0(t)}{a^2 + t^2} dt + \int_{t^*}^{\infty} \frac{U_0(t)}{a^2 + t^2} dt \right).$$

When $k \neq 0$ we write

$$\left(\int_{-\infty}^{-t^*} + \int_{t^*}^{-\frac{1}{2}} + \int_{-\frac{1}{2}}^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{t^*} + \int_{t^*}^{\infty} \right) \frac{U_0(t)}{a^2 + (b-t)^2} dt = I_1 + I_2 + I_3 + I_4 + I_5,$$

say. We may estimate each of I_2 , I_3 , and I_4 numerically, noting that the formula used for $U_0(t)$ is different in each of these integrals. For $|t| \geq t^*$, we find that

$$U_0(|t|) \leq \log |t| - c,$$

and the optimal value of c is

$$c = \log 2 + \frac{2}{1+4t^{*2}} - \frac{2}{3t^*} - \frac{1}{8t^{*2}} = 0.462935\dots$$

Thus, we obtain

$$\begin{aligned} I_1 &= \int_{-\infty}^{-t^*} \frac{U_0(t)}{a^2 + (b-t)^2} dt < \int_{-\infty}^{-t^*} \frac{\log(-t) - c}{(b-t)^2} dt \\ &= \frac{\log(b+t^*)}{b} - \frac{t^* \log t^*}{b(b+t^*)} - \frac{c}{b+t^*}. \end{aligned}$$

All that remains is to estimate I_5 . First, we note that

$$\int_{t^*}^{\infty} \frac{c}{a^2 + (b-t)^2} dt = \frac{c}{a} \left(\tan^{-1} \left(\frac{b-t^*}{a} \right) + \frac{\pi}{2} \right).$$

We then divide the range into three sections: $[t^*, b(1 - \varepsilon)]$, $[b(1 - \varepsilon), b(1 + \varepsilon)]$, and $[b(1 + \varepsilon), \infty)$, for a parameter $\varepsilon > 0$. We compute

$$\begin{aligned} \int_{t^*}^{b(1-\varepsilon)} \frac{\log t}{a^2 + (b-t)^2} dt &< \int_{t^*}^{b(1-\varepsilon)} \frac{\log t}{(b-t)^2} dt \\ &= \frac{1}{b} \left(\frac{\log b}{\varepsilon} - \log(b-t^*) + \log \varepsilon - \left(1 - \frac{1}{\varepsilon}\right) \log(1-\varepsilon) - \frac{t^* \log t^*}{b-t^*} \right), \end{aligned}$$

and then

$$\int_{b(1-\varepsilon)}^{b(1+\varepsilon)} \frac{\log t}{a^2 + (b-t)^2} dt < \log(b(1+\varepsilon)) \cdot \frac{2}{a} \tan^{-1} \left(\frac{b\varepsilon}{a} \right) < \frac{\pi}{a} \log(b(1+\varepsilon)),$$

and finally

$$\int_{b(1+\varepsilon)}^{\infty} \frac{\log t}{a^2 + (b-t)^2} dt < \int_{b(1+\varepsilon)}^{\infty} \frac{\log t}{(b-t)^2} dt = \frac{1}{b\varepsilon} \left(\log b + \varepsilon \log \left(1 + \frac{1}{\varepsilon}\right) + \log(1+\varepsilon) \right).$$

Therefore, we find that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{U_0(t)}{a^2 + (b-t)^2} dt &< \int_{-t^*}^{-\frac{1}{2}} \frac{U_0(t)}{a^2 + (b-t)^2} dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{U_0(t)}{a^2 + (b-t)^2} dt \\ &+ \int_{\frac{1}{2}}^{t^*} \frac{U_0(t)}{a^2 + (b-t)^2} dt + \frac{\pi}{a} \log(b(1+\varepsilon)) \\ &+ \frac{1}{b} \left(\frac{2 \log b}{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right) \log(1+\varepsilon) - \left(1 - \frac{1}{\varepsilon}\right) \log(1-\varepsilon) + \log \left(\frac{b+t^*}{b-t^*} \right) \right) \\ &- \frac{t^* \log t^*}{b} \left(\frac{1}{b+t^*} + \frac{1}{b-t^*} \right) - c \left(\frac{1}{b+t^*} + \frac{1}{a} \left(\tan^{-1} \left(\frac{b-t^*}{a} \right) + \frac{\pi}{2} \right) \right). \end{aligned}$$

We choose $\varepsilon = 10^{-3}$, and thus find that we can bound the term $C_{41}(k)$ relatively easily.

5. SEARCHING FOR TRIGONOMETRIC POLYNOMIALS

We employ simulated annealing to search for favorable trigonometric polynomials in this problem. In this optimization method, one seeks to minimize an objective function G defined on some (typically multidimensional) domain D . One begins at a particular location, and moves through the space in search of favorable values of G by using an iterative process. At each step, one moves from the current location to a nearby point in the domain D , and measures the change in the value of G . If the objective function decreased, i.e. we witnessed an improvement, then we keep this step and proceed to the next iteration. On the other hand, if we find that G increased due to this move, then we may undo this step, but we may choose to keep it, depending on a particular probability distribution. This distribution depends on a parameter called the *temperature*: if the change ΔG in the objective function is positive, then we keep this step with probability $\exp(-\Delta G/Y)$, where Y is the current temperature. Thus, if the temperature is high, then we are reasonably likely to keep steps that increase the objective function by a small amount, but less likely if we witness a large change. As the temperature decreases, we become less likely to accept small increases in the value of G . In the limiting case, as $Y \rightarrow 0^+$, the method becomes simple greedy descent. Typically, one performs a number of trials

for each value in a decreasing sequence of temperatures (the *annealing schedule*), ending with a number of trials of simple greedy descent.

Simulated annealing requires many evaluations of the objective function, so it is very helpful if this function can be computed quickly. In this problem, where the domain D is the set P_n of trigonometric polynomials of degree n having the required properties, using Kadiri's method to evaluate the merit of a particular polynomial in this space is rather time-consuming. Instead, we use Landau's function from (1.1), so for $f \in P_n$ we let

$$G(f) = \frac{A}{a_0 + a_1 - 2\sqrt{a_0 a_1}}, \quad (5.1)$$

where A again denotes $f(0) - a_0$. Note that writing the denominator in this way allows G to be evaluated with only one call to a square-root function.

In our method, we fix the degree n , we set $c_0 = 1$, and then choose each c_k for $1 \leq k \leq n$ uniformly at random from $[0, B]$ for a prescribed bound B , and then set

$$f(\varphi) = \left| \sum_{k=0}^n c_k e^{ik\varphi} \right|^2.$$

We then compute values a_0, \dots, a_n so that $f(\varphi) = \sum_{k=0}^n a_k \cos(k\varphi)$. Here,

$$a_0 = \sum_{j=0}^n c_j^2, \quad a_k = 2 \sum_{j=0}^{n-k} c_j c_{j+k} \text{ for } 1 \leq k \leq n.$$

We note that a_0 is the peak autocorrelation of the sequence $\{c_k\}$, and $a_k/2$ for $k > 1$ is the k th off-peak aperiodic autocorrelation of this sequence. (In fact, it is slightly more efficient to keep track of these autocorrelations in the code, rather than the a_k , but this is a minor point.) We then test if $f \in P_n$, so we test if $a_1 > a_0$ and whether each $a_k \geq 0$, since $f(\varphi) \geq 0$ for all φ automatically by the construction. (Kondrat'ev [6] employed this same strategy to ensure choosing trigonometric polynomials that are nonnegative everywhere.) If this test fails, then we start over with a new randomly selected f . Once we have a viable polynomial, we proceed with the annealing phase.

To describe the annealing process, we find it convenient to set $Z = 1/Y$, where Y is the temperature. We begin with a particular maximum step size S and an initial value for Z . We then perform M annealing trials: each time we randomly select an integer $k \in \{1, \dots, n\}$, we pick a step size $s \in [-S, S]$ uniformly at random, and then we change c_k by s . We then update each coefficient a_j that depends on c_k , so a_0 increases by $s(2c_k + s)$, a_k increases by $2s$, each a_i with $1 \leq i < k$ increases by $2sc_{k-i}$, and each a_i with $1 \leq i < n - k$ increases by $2sc_{k+i}$. Thus, each change to a value of c_k requires $O(n)$ operations to update the state of the program. We then check if the new values of a_i are all nonnegative, and that $a_1 > a_0$. If any of these conditions fails, we undo this step and return to the prior polynomial f . Otherwise, our new location is valid, so we compute the value of the objective function (5.1). If $\Delta G < 0$, we keep this step, and if $\Delta G > 0$, then we keep this step with probability $e^{-Z\Delta G}$.

For each fixed value of the maximum step size S , we perform this process M times for each of the K values of Z , incrementing Z each time by an amount ΔZ , and then we perform M trials of greedy descent. After this, we decrease the step

FIGURE 1. Coefficients $\{c_k\}$ for $F_{40}(\varphi)$.

```
{1, 5.82299804516981, 20.256046857268, 41.8371543572416, 62.6803646661328,
86.3371461181984, 140.237920299854, 230.172136828422, 296.566863709684,
259.103949548556, 115.007833826561, -36.3610468402636, -95.0264057388131,
-58.142464967588, -2.7847512808475, 17.2100840100492, 10.7193028056306,
5.04620845731704, 3.86024903930945, -0.321719428284046, -4.25534268200327,
-2.73295949107489, 1.31013188826865, 1.16950963394944, -0.36353468679768,
-0.468164164916908, 0.940864822848766, 0.224723973462492, -0.762490754252658,
-0.100627491842387, 0.486777412937381, -0.00593275691352972, -0.399968301710688,
0.225086077352437, 0.198770260275958, -0.239235383947936, -0.0483116501842605,
0.212556439828151, -0.124475686268429, 0.0123641022008911, 0.0146979855952472}
```

size by dividing S by $1 + \lambda$, where λ is a positive parameter, and continue as long as this maximum size is greater than a prescribed value, S_0 .

Our selection of parameters in this process varied over a number of runs. In fact, it was useful to run annealing jobs on many processors simultaneously, with each processor choosing its annealing parameters at random from some prescribed intervals. This way, over time we could learn which parameters tended to produce better trigonometric polynomials, and this would inform choices of parameters in later runs. We often chose $B \in [100, 200]$, $Z \in [8, 16]$, $\Delta Z \in [.5, 2]$, $M \in [250n, 350n]$, $K = 10$ or 11 , $S \in [2.5, 4]$, $\lambda \in [.015, .05]$, and $S_0 = 10^{-5}$ or 10^{-6} .

6. RESULTS

Table 4 records the best value of the objective function we constructed by using this optimization method for each degree $n = 4m$, with $2 \leq m \leq 10$. Each entry provides the current best-known value for Landau's V_n (1.1). In particular, the entry for $n = 8$ improves the result of [6], and each subsequent entry is less than the formerly best known upper bound on V of $34.5035\dots$ from [1]. At degree $n = 40$, our searches produced a polynomial that established $V_{40} < 34.48923$. We then performed some additional simulated annealing passes with finer parameters, using this polynomial as our starting point. In this way we constructed a trigonometric polynomial F_{40} , whose objective value produces $V_{40} < 34.488992000856$. This establishes Theorem 2. The coefficient sequence $\{c_k\}$ for $F_{40}(\varphi)$ is shown in Figure 1, and the corresponding values for the sequence $\{a_k\}$, after normalizing by dividing by the sum of the squares of the c_k (approximately 268 761.1), are displayed in Figure 2. The sum of all the coefficients after a_0 in this list is $A = 3.490002852278399$. A plot for $F_{40}(\varphi)$ over $[\pi/2, \pi]$ is exhibited in Figure 3.

We may now apply Kadiri's method for computing a value for R_0 in the classical zero-free region of the Riemann zeta-function using each of the polynomials that produced the records for V_n , and employing the savings obtained in the error term as described in Section 3. These values are also listed in Table 4, along with the value of θ that was employed in each case. The best value here appears at $n = 12$, and some further local optimizations allow us to derive $R_0 < 5.57422$. It is interesting that polynomials that produce an improvement in Landau's value V do not necessarily produce an improvement in R_0 under Kadiri's method, but this is perhaps not so surprising since we use Landau's function as a surrogate for our true objective function, owing to its easy calculation. This suggests that

FIGURE 2. Normalized coefficients $\{a_k\}$ for $F_{40}(\varphi)$.

$\{1, 1.737404932358421, 1.1180312174988238, 0.4958068290777618, 0.12043435038423943,$
 $2.8862586376137162 \cdot 10^{-15}, 2.0972266792962233 \cdot 10^{-7}, 0.010264161325212823,$
 $0.005445447694684562, 4.982862012828773 \cdot 10^{-14}, 4.8092823793119544 \cdot 10^{-8},$
 $6.247326172856939 \cdot 10^{-4}, 1.625564268382144 \cdot 10^{-12}, 2.835025226384611 \cdot 10^{-7},$
 $6.129588083976596 \cdot 10^{-4}, 5.231422915759923 \cdot 10^{-4}, 4.936509000300511 \cdot 10^{-8},$
 $1.4092336429708759 \cdot 10^{-13}, 2.923444923088646 \cdot 10^{-4}, 2.3484834215069657 \cdot 10^{-4},$
 $1.1342395284997809 \cdot 10^{-7}, 9.170949177475853 \cdot 10^{-15}, 1.1678447544116879 \cdot 10^{-4},$
 $8.482954015115613 \cdot 10^{-5}, 5.338369918854597 \cdot 10^{-12}, 1.8115431364172399 \cdot 10^{-6},$
 $2.274822660800113 \cdot 10^{-5}, 6.894175279654467 \cdot 10^{-14}, 1.570310005695605 \cdot 10^{-7},$
 $3.090195669976266 \cdot 10^{-5}, 2.75304675142816 \cdot 10^{-5}, 4.959939074564098 \cdot 10^{-8},$
 $2.6604018245602642 \cdot 10^{-14}, 1.643513306111948 \cdot 10^{-5}, 1.522818206748077 \cdot 10^{-5},$
 $4.623133724479399 \cdot 10^{-6}, 7.930664147911865 \cdot 10^{-7}, 2.6276458478038436 \cdot 10^{-6},$
 $1.8249928742340992 \cdot 10^{-6}, 7.289034419080702 \cdot 10^{-7}, 1.0937584248561976 \cdot 10^{-7}\}$

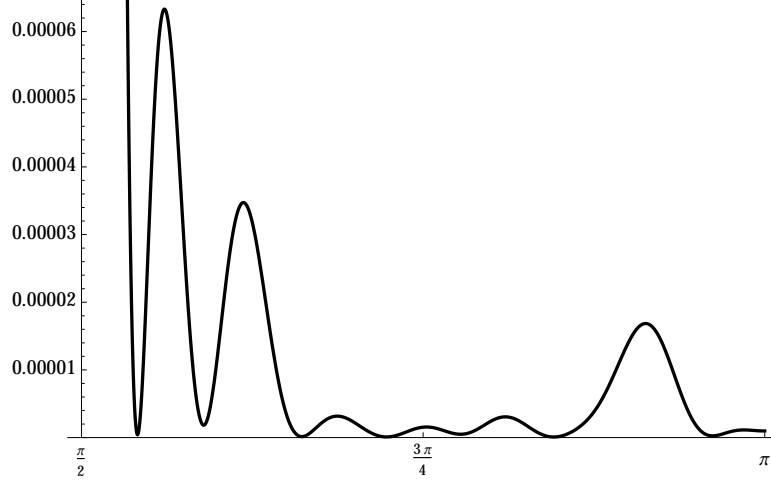
TABLE 4. Upper bound on V_n for $n = 4m$ and $2 \leq m \leq 10$, and bound on R_0 resulting from choosing this polynomial.

n	$V_n <$	θ	$R_0 <$
8	34.53991919	1.853	5.58139
12	34.50266054	1.855	5.57429
16	34.49747009	1.855	5.57490
20	34.49321564	1.855	5.57495
24	34.49027559	1.855	5.57519
28	34.48959029	1.855	5.57560
32	34.48939230	1.855	5.57632
36	34.48930967	1.855	5.57683
40	34.488992001	1.855	5.57724

we test the several hundred thousand other trigonometric polynomials that were produced by our method of simulated annealing, which exhibited very good values under Landau's objective function. Indeed, many of these produce a value for R_0 better than 5.57422. The best polynomial found in these searches has degree 16, and produces $R_0 < 5.57353$. Some further annealing using this polynomial as a starting point produces the polynomial $F_{16}(\varphi)$ employed in Section 3 to establish the bound in Theorem 1. The sequences of coefficients c_k and a_k are both displayed in Table 5, where again we have normalized the a_k by dividing by the sum of the squares of the c_k (approximately 932 023.9), so that $a_0 = 1$.

We remark that Landau's objective function (5.1) evaluated on $F_{16}(\varphi)$ produces 34.49997..., which is somewhat larger than the record obtained for this degree, as shown in Table 4.

6.1. Potential improvements. While it appears that the scope for selecting an improved trigonometric polynomial may be limited, we see that the height T_0 to which the Riemann hypothesis has been verified has some influence on the width of the zero-free region that one can obtain by using this method. We might ask then

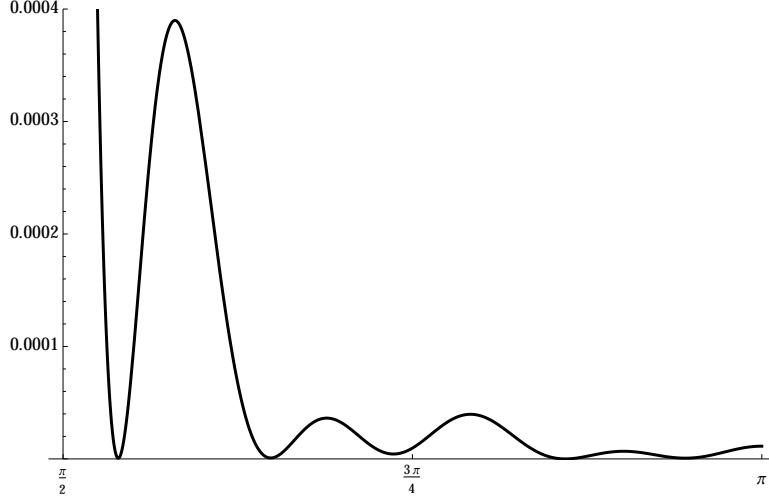
FIGURE 3. $F_{40}(\varphi)$ for $\pi/2 \leq \varphi \leq \pi$.TABLE 5. Coefficients for $F_{16}(\varphi)$.

c_0	1	a_0	1
c_1	-2.09100370089199	a_1	1.74126664022806
c_2	0.414661861733616	a_2	1.128282822804652
c_3	4.94973437766435	a_3	0.5065272432186642
c_4	-2.26052224951171	a_4	0.1253566902628852
c_5	-8.58599241204357	a_5	$9.35696526707405 \cdot 10^{-13}$
c_6	6.87053689828658	a_6	$4.546614790384321 \cdot 10^{-13}$
c_7	22.6412990090005	a_7	0.01201214561729989
c_8	-6.76222005424994	a_8	0.006875849760911001
c_9	-50.2233943767588	a_9	$7.77030543093611 \cdot 10^{-12}$
c_{10}	8.07550113395201	a_{10}	$2.846662294985367 \cdot 10^{-7}$
c_{11}	223.771572768515	a_{11}	0.001608306592372963
c_{12}	487.278135806977	a_{12}	0.001017994683287104
c_{13}	597.268928658734	a_{13}	$2.838909054508971 \cdot 10^{-7}$
c_{14}	473.937203439807	a_{14}	$5.482482041999887 \cdot 10^{-6}$
c_{15}	237.271715181426	a_{15}	$2.412958794855076 \cdot 10^{-4}$
c_{16}	59.6961898512813	a_{16}	$1.281001290654868 \cdot 10^{-4}$
		A	3.523323140225021

how much improvement we might expect in R_0 from this method as the parameter T_0 increases. For example, if $T_0 = 3 \cdot 10^{11}$, then by choosing $\theta = 1.85567$ with $F_{16}(\varphi)$, we obtain that $R_0 = 5.5666305$ is permissible.¹

We also applied this method for 150 values for T_0 up to approximately 10^{300} , using $F_{16}(\varphi)$ as the trigonometric polynomial, with $t_0 = 10^5$, $r = 5$, and $\theta =$

¹The value $T_0 = 3 \cdot 10^{11}$ has been announced by Jan Bütke and Jens Franke in a personal communication.

FIGURE 4. $F_{16}(\varphi)$ for $\pi/2 \leq \varphi \leq \pi$.

1.8552. We found that the value of R_0 that one may obtain in this region is well-approximated by

$$R_0 = 5.4912 + \frac{2.0185}{\log T_0},$$

suggesting that, as larger values of T_0 are established, the potential improvement in the width of the classical zero-free region that one may obtain by using this method is limited.

7. APPLICATIONS

Theorem 1 has a number of applications. We first describe two improvements concerning the error term in the prime number theorem that follow almost immediately from this result.

Corollary 1. *Let*

$$\epsilon_0(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} e^{-X}, \quad X = \sqrt{(\log x)/R}, \quad R = 6.315.$$

Then

$$\begin{aligned} |\theta(x) - x| &\leq x\epsilon_0(x), & x &\geq 149, \\ |\psi(x) - x| &\leq x\epsilon_0(x), & x &\geq 23. \end{aligned}$$

Proof. Theorem 1 can be used, as in [19, p. 2], to show that there are no zeros of the Riemann zeta-function in the region

$$\sigma \geq 1 - \frac{1}{6.315 \log |t/17|}, \quad t \geq 24. \quad (7.1)$$

The statement then follows from the arguments in [19]. \square

Here, the value 6.315 replaces the value 6.455 from [19]. The same substitution occurs in the following improvement on a result from this same article, where in addition the leading constant term is sharpened here from 0.2795 to 0.2593.

Corollary 2. *If $x \geq 229$ then*

$$|\pi(x) - \text{li}(x)| \leq 0.2593 \frac{x}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{6.315}}\right).$$

Proof. The proof is almost identical to that in [19]. We make use of the more sophisticated result from [11] that $\theta(t) < t$ for all $t < 1.39 \times 10^{17}$, and the proof follows as in [19] by choosing $\alpha = 1.70$ and $x_0 = 1.5 \times 10^8$. \square

It is worthwhile to note that a zero-free region of the form

$$\sigma \geq 1 - \frac{1}{R \log(t/B)}, \quad (7.2)$$

for $t \geq 2$, where $B \geq 1$ is a fixed constant, is asymptotically equivalent to the form analyzed in this paper. Were one to prove (7.2) for some $B > 1$ with $R = R_0$, for the value of R_0 derived in this paper, then one could further improve on Corollaries 1 and 2.

Finally, we very briefly note two additional applications of Theorem 1. First, Faber and Kadiri [3], continuing the work of Schoenfeld [17], have used the width of this zero-free region *inter alia* to obtain good bounds for the Chebyshev functions $\theta(x) = \sum_{p \leq x} \log p$ and $\psi(x) = \sum_{n \leq x} \Lambda(n)$, where $\Lambda(n)$ is von Mangoldt's function. Second, Ramaré [12, Thm. 1.1] employs Kadiri's value of R_0 when bounding $\sum_{n \leq x} \Lambda(n)/n$. These results may be updated by using the smaller value of R_0 from Theorem 1.

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